

# A family of surfaces with $p_g = q = 2$ , $K^2 = 7$ and Albanese map of degree 3

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## Abstract

We study a family of surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$ , originally constructed by Cancian and Frapporti by using the Computer Algebra System MAGMA. We provide an alternative, computer-free construction of these surfaces, that allows us to describe their Albanese map and their moduli space.

## 0 Introduction

In recent years, the work of several authors on the classification of *irregular* algebraic surfaces (that is, surfaces  $S$  with  $q(S) > 0$ ) produced a considerable amount of results, see for example the survey papers [BaCaPi06, MP12] for a detailed bibliography on the subject.

In particular, surfaces of general type with  $\chi(\mathcal{O}_S) = 1$ , that is,  $p_g(S) = q(S)$  were investigated. For these surfaces, [Deb81, Théorème 6.1] implies  $p_g \leq 4$ . Surfaces with  $p_g = q = 4$  and  $p_g = q = 3$  are nowadays completely classified, see [Be82, CaCiML98, HP02, Pir02]. On the other hand, for the the case  $p_g = q = 2$ , which presents a very rich and subtle geometry, we have so far only a partial understanding of the situation; we refer the reader to [PePol13a, PePol13b, PePol14] for an account on this topic and recent results.

As the title suggest, in this paper we consider a family  $\mathcal{M}$  of minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$ . The existence of such surfaces was originally established in [CanFr15] with the help of the Computer Algebra System MAGMA [BoCaPi97]; the present work provides an alternative, computer-free construction of them, that allows us to describe their Albanese map and their moduli space.

Our results can be summarized as follows, see Theorem 2.7.

**Main Theorem.** *There exists a 3-dimensional family  $\mathcal{M}$  of minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$  such that, for all elements  $S \in \mathcal{M}$ , the canonical class  $K_S$  is ample and the Albanese map  $\alpha: S \rightarrow A$  is a generically finite triple cover of a principally polarized abelian surface  $(A, \Theta)$ , simply branched over a curve  $D_A$  numerically equivalent to  $4\Theta$  having an ordinary sextuple point and no other singularities. The family  $\mathcal{M}$  provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space  $\mathcal{M}_{2,2,7}^{\text{can}}$  of canonical models of minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$ .*

In particular, this means that  $\mathcal{M}$  provides a dense open set of a generically smooth, irreducible component of  $\mathcal{M}_{2,2,7}^{\text{can}}$ . Furthermore, denoting by  $\mathcal{M}_2$  the coarse moduli space

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of curves of genus 2, there exists a quasi-finite, surjective morphism  $\varsigma: \mathcal{M} \rightarrow \mathcal{M}_2$  of degree 40 (see Proposition 2.9).

Let us explain now how the paper is organized. In Section 1 we explain our construction in detail and we compute the invariants of the resulting surfaces (Proposition 1.5); moreover we study their Albanese map, giving a precise description of its image and of its branch curve (Proposition 1.8). It is worth pointing out that the general surface  $S$  contains no irrational pencils (Proposition 1.9).

Section 2 is devoted to the study of the first-order deformations of the surfaces in  $\mathcal{M}$  and to the description of the corresponding subset in  $\mathcal{M}_{2,2,7}^{\text{can}}$ . A key point in our analysis is showing that for all elements in  $S \in \mathcal{M}$  we have  $h^1(S, T_S) = 3$ , see Proposition 2.6.

Since the degree of the Albanese map is in this case a topological invariant (Proposition 2.1), it follows that these surfaces lie in a different connected component of the moduli space than the only other known example with the same invariants, namely the surface with  $p_g = q = 2$  and  $K^2 = 7$  constructed in [Ri15], whose Albanese map is a generically finite *double* cover of an abelian surface with polarization of type  $(1, 2)$ , see Remark 2.8. Hence the family  $\mathcal{M}$  provides a substantially new piece in the fine classification of minimal surfaces of general type with  $p_g = q = 2$ .

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**Notation and conventions.** We work over the field  $\mathbb{C}$  of complex numbers. By *surface* we mean a projective, non-singular surface  $S$ , and for such a surface  $K_S$  denotes the canonical class,  $p_g(S) = h^0(S, K_S)$  is the *geometric genus*,  $q(S) = h^1(S, K_S)$  is the *irregularity* and  $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$  is the *Euler-Poincaré characteristic*.

If  $C$  is a smooth curve, we identify  $\text{Pic}^0(C)$  with the Jacobian variety  $J(C)$  by means of the canonical isomorphism provided by the Abel-Jacobi map, see [BL94, Theorem 11.1.3]. Furthermore, we write  $\text{Sym}^n(C)$  for the  $n$ -th symmetric product of  $C$ .

Given a finite group  $G$  acting on a vector space  $V$ , we denote by  $V^G$  the  $G$ -invariant subspace.

## 1 The construction

Let  $V_2$  and  $V_3$  be the two hypersurfaces of  $\mathbb{P}^3$  defined by

$$V_2 := \{x_2x_3 + r(x_0, x_1) = 0\}, \quad V_3 := \{x_2^3 + x_3^3 + s(x_0, x_1) = 0\}, \quad (1)$$

where  $r, s \in \mathbb{C}[x_0, x_1]$  are general homogeneous forms of degree 2 and 3, respectively. Then  $C_4 := V_2 \cap V_3$  is a smooth, canonical curve of genus 4. Denoting by  $\xi$  a primitive

third root of unity, we see that  $C_4$  admits a free action of the cyclic group  $\langle \xi \rangle \cong \mathbb{Z}/3\mathbb{Z}$ , defined by

$$\xi \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : \xi x_2 : \xi^2 x_3] \quad (2)$$

and the quotient  $C_2 := C_4/\langle \xi \rangle$  is a smooth curve of genus 2.

**Proposition 1.1.** *All étale Galois triple covers of a smooth curve of genus 2 can be obtained in this way.*

*Proof.* Let  $C_2$  be any smooth curve of genus 2 and choose any étale  $\mathbb{Z}/3\mathbb{Z}$ -cover  $c: C_4 \rightarrow C_2$ . Thus  $C_4$  is a smooth curve of genus 4 and we can choose a fixed-point free automorphism  $\varphi: C_4 \rightarrow C_4$  generating the Galois group of the cover.

The curve  $C_4$  cannot be hyperelliptic, otherwise its ten Weierstrass points would be an invariant set by any automorphism, which is impossible because any orbit of  $c$  consists of three distinct points. Hence the canonical divisor  $K_{C_4}$  is very ample and defines an embedding of  $C_4$  in  $\mathbb{P}^3 = \mathbb{P}H^0(C_4, K_{C_4})$ , whose image (that we still denote by  $C_4$ ) is the complete intersection of a (uniquely determined) quadric hypersurface  $V_2$  and a cubic hypersurface  $V_3$ . It remains to show that we can choose  $V_2$  and  $V_3$  as in (1).

Pushing down the canonical line bundle of  $C_4$  to  $C_2$  gives a decomposition of  $H^0(C_4, K_{C_4})$  into  $\mathbb{Z}/3\mathbb{Z}$ -eigenspaces, namely

$$H^0(C_4, K_{C_4}) = H^0(C_2, K_{C_2}) \oplus H^0(C_2, K_{C_2} + \eta) \oplus H^0(C_2, K_{C_2} + 2\eta) \quad (3)$$

where  $\eta$  is a non-trivial, 3-torsion divisor on  $C_2$ . The first summand in (3) has dimension 2, whereas the others have dimension 1; so we can choose a basis  $x_0, x_1, x_2, x_3$  of  $H^0(C_4, K_{C_4})$  such that  $x_0, x_1$  generate  $H^0(C_2, K_{C_2})$  whereas  $x_2$  and  $x_3$  generate  $H^0(C_2, K_{C_2} + \eta)$  and  $H^0(C_2, K_{C_2} + 2\eta)$ , respectively. This means that, using homogeneous coordinates  $[x_0 : x_1 : x_2 : x_3]$  in  $\mathbb{P}^3$ , the action of  $\mathbb{Z}/3\mathbb{Z} = \langle \xi \rangle$  can be written as in (2).

We start by looking at the invariant quadrics in the homogeneous ideal of  $C_4$ . There are four invariant monomials of degree 2, namely

$$x_0^2, x_0x_1, x_1^2, x_2x_3, \quad (4)$$

hence the invariant subspace  $(\text{Sym}^2 H^0(C_4, K_{C_4}))^{(\xi)}$  of  $\text{Sym}^2 H^0(C_4, K_{C_4})$  has dimension 4. On the other hand, the subspace of invariant quadrics in the homogeneous ideal of  $C_4$  is the kernel of the surjective map

$$(\text{Sym}^2 H^0(C_4, K_{C_4}))^{(\xi)} \rightarrow H^0(C_4, 2K_{C_4})^{(\xi)} \cong H^0(C_2, 2K_{C_2}) \cong \mathbb{C}^3,$$

hence it has dimension 1. In other words, the unique quadric  $V_2$  containing  $C_4$  is invariant, hence the polynomial defining  $V_2$  is a linear combination of the monomials in (4). The coefficient of  $x_2x_3$  cannot vanish, or  $V_2$  would be reducible, so  $V_2$  is as in (1).

Let us look now at the invariant cubics in the homogeneous ideal of  $C_4$ . There are eight invariant monomials of degree 3, namely

$$x_0^3, x_0^2x_1, x_0x_1^2, x_1^3, x_0x_2x_3, x_1x_2x_3, x_2^3, x_3^3,$$

hence the invariant subspace  $(\text{Sym}^3 H^0(C_4, K_{C_4}))^{(\xi)}$  of  $\text{Sym}^3 H^0(C_4, K_{C_4})$  has dimension 8. On the other hand, the subspace of invariant cubics in the homogeneous ideal of  $C_4$  is the kernel of the surjective map

$$(\text{Sym}^3 H^0(C_4, K_{C_4}))^{(\xi)} \rightarrow H^0(C_4, 3K_{C_4})^{(\xi)} \cong H^0(C_2, 3K_{C_2}) \cong \mathbb{C}^5,$$

hence it has dimension 3. In particular, this implies that the general invariant cubic hypersurface  $V_3$  containing  $C_4$  is not a multiple of the quadric  $V_2$ . Adding suitable scalar multiples of  $x_0V_2$  and  $x_1V_2$  in order to get rid of the monomials  $x_0x_2x_3$  and  $x_1x_2x_3$ , and changing coordinates by multiplying  $x_2$  and  $x_3$  by suitable constants we obtain an equation of  $V_3$  as in (1) and we are done.  $\square$

Let us consider now the product  $C_4 \times C_4 \subset \mathbb{P}^3 \times \mathbb{P}^3$ , and write  $\mathbf{x} = [x_0 : x_1 : x_2 : x_3]$  for the homogeneous coordinates in the first factor and  $\mathbf{y} = [y_0 : y_1 : y_2 : y_3]$  for those in the second factor. Then the action of  $\langle \xi \rangle$  on  $C_4$  induces an action of  $H := \langle \xi_x, \xi_y, \sigma \rangle$  on  $C_4 \times C_4$ , where

$$\xi_x(\mathbf{x}, \mathbf{y}) := (\xi \cdot \mathbf{x}, \mathbf{y}), \quad \xi_y(\mathbf{x}, \mathbf{y}) := (\mathbf{x}, \xi \cdot \mathbf{y}), \quad \sigma(\mathbf{x}, \mathbf{y}) := (\mathbf{y}, \mathbf{x}).$$

Clearly  $\xi_x$  and  $\xi_y$  commute, whereas  $\sigma\xi_x = \xi_y\sigma$  and  $\sigma\xi_y = \xi_x\sigma$ , so  $H$  is a semi-direct product of the form

$$H = \langle \xi_x, \xi_y \rangle \rtimes \langle \sigma \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}.$$

In particular,  $|H| = 18$  and every element  $h \in H$  can be written in a unique way as  $h = \sigma^k \xi_x^i \xi_y^j$ , where  $k \in \{0, 1\}$  and  $i, j \in \{0, 1, 2\}$ .

**Lemma 1.2.** *The non-trivial elements of  $H$  having fixed points on  $C_4 \times C_4$  are precisely the three elements of order 2*

$$h_i := \sigma \xi_x^i \xi_y^{3-i}, \quad i = 0, 1, 2.$$

More precisely, the element  $h_i$  fixes pointwise the smooth curve

$$\Gamma_i := \{(\mathbf{x}, \xi^i \cdot \mathbf{x}) \mid \mathbf{x} \in C_4\},$$

that is, the graph of the automorphism of  $C_4$  defined by  $\mathbf{x} \mapsto \xi^i \cdot \mathbf{x}$ . The three curves  $\Gamma_0$ ,  $\Gamma_1$  and  $\Gamma_2$  are isomorphic to  $C_4$ , pairwise disjoint and their self-intersection equals  $-6$ .

*Proof.* Let  $h = \sigma^k \xi_x^i \xi_y^j$  be an element of  $H$ . If  $k = 0$  then  $h(\mathbf{x}, \mathbf{y}) = (\xi^i \cdot \mathbf{x}, \xi^j \cdot \mathbf{y})$  so, since the action of  $\xi$  on  $C_4$  is free,  $h$  has fixed points if and only if it is trivial. Thus we can assume  $k = 1$ , in which case we have

$$\sigma \xi_x^i \xi_y^j(\mathbf{x}, \mathbf{y}) = (\xi^j \cdot \mathbf{y}, \xi^i \cdot \mathbf{x}).$$

Hence  $(\mathbf{x}, \mathbf{y})$  is a fixed point for  $h$  if and only if  $i + j \equiv 0 \pmod{3}$  and  $\mathbf{y} = \xi^i \cdot \mathbf{x}$ , that is  $(\mathbf{x}, \mathbf{y}) \in \Gamma_i$ .

A straightforward computation using the relations  $\sigma^2 = 1$  and  $\xi_x\sigma = \sigma\xi_y$  shows that the order of  $h_i$  is 2.

The curve  $\Gamma_0$  is the diagonal of  $C_4 \times C_4$ , hence it is isomorphic to  $C_4$  and satisfies  $(\Gamma_0)^2 = 2 - 2g(C_4) = -6$ . The same is true for the curves  $\Gamma_1$  and  $\Gamma_2$ , because they are the translate of  $\Gamma_0$  by the action of  $\xi_y$  and  $\xi_x$ , respectively. Finally,  $\Gamma_i$  and  $\Gamma_j$  are disjoint for  $i \neq j$ , because  $\xi$  acts freely on  $C_4$ .  $\square$

Lemma 1.2 implies that the quotient map  $C_4 \times C_4 \rightarrow (C_4 \times C_4)/H$  is ramified exactly over the three curves  $\Gamma_i$ , with ramification index 2 on each of them. We factor such a map through the quotient by the normal abelian subgroup  $\langle \xi_x, \xi_y \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2$ . This subgroup acts separately on the two factors, whereas  $\sigma$  exchanges them, so we get

$$(C_4 \times C_4)/\langle \xi_x, \xi_y \rangle \cong C_2 \times C_2, \quad (C_4 \times C_4)/H \cong \text{Sym}^2(C_2).$$

Thus the surface  $B = (C_4 \times C_4)/H$  contains a unique rational curve, namely the  $(-1)$ -curve  $E$  corresponding to the unique  $g_2^1$  of  $C_2$ . Denoting by  $\pi: B \rightarrow A$  the blow-down of  $E$ , we see that  $A$  is an abelian surface isomorphic to the Jacobian variety  $J(C_2)$ .

**Remark 1.3.** Because of Proposition 1.1, all Jacobians of smooth curves of genus 2 can be obtained in this way.

Let us denote now by  $\xi_{xy}$  the element  $\xi_x \xi_y$  and set  $G := \langle \xi_{xy}, \sigma \rangle$ ; then  $G$  is a non-normal, abelian subgroup of  $H$ , isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . Setting

$$T := (C_4 \times C_4) / \langle \xi_{xy} \rangle, \quad S := (C_4 \times C_4) / G,$$

and writing  $t: C_4 \times C_4 \rightarrow T$  and  $f: C_4 \times C_4 \rightarrow S$  for the corresponding projection maps, we have the following commutative diagram:

$$\begin{array}{ccc}
 C_4 \times C_4 & \xrightarrow{\quad t \quad} & T = (C_4 \times C_4) / \langle \xi_{xy} \rangle \\
 & \searrow f & \downarrow u \\
 & & S = (C_4 \times C_4) / G
 \end{array}
 \quad
 \begin{array}{ccc}
 & \xrightarrow{\quad \gamma \quad} & (C_4 \times C_4) / \langle \xi_x, \xi_y \rangle \cong C_2 \times C_2 \\
 & & \downarrow v \\
 & \xrightarrow{\quad \beta \quad} & (C_4 \times C_4) / H = B \cong \text{Sym}^2(C_2) \\
 & \searrow \alpha & \downarrow \pi \\
 & & A.
 \end{array}
 \tag{5}$$

The morphism  $u: T \rightarrow S$  is a double cover, induced by the involution  $\sigma$  exchanging the two coordinates in  $C_4 \times C_4$ .

We first compute the invariants of  $T$ .

**Lemma 1.4.** *The surface  $T$  is minimal of general type with*

$$p_g(T) = 6, \quad q(T) = 4, \quad K_T^2 = 24.$$

*Proof.* By standard calculations we have

$$p_g(C_4 \times C_4) = 16, \quad q(C_4 \times C_4) = 8, \quad K_{C_4 \times C_4}^2 = 72.$$

The group  $\langle \xi_{xy} \rangle \cong \mathbb{Z}/3\mathbb{Z}$  acts diagonally and freely on  $C_4 \times C_4$ , hence  $T$  is a so-called *quasi-bundle*, see for instance [Pol09, Section 3]. Therefore we obtain

$$K_T^2 = \frac{1}{3} K_{C_4 \times C_4}^2 = 24, \quad \chi(\mathcal{O}_T) = \frac{1}{3} \chi(\mathcal{O}_{C_4 \times C_4}) = 3, \quad q(T) = g(C_2) + g(C_2) = 4,$$

so  $p_g(T) = 6$ . Note that by Noether's formula this implies  $c_2(T) = 12$ . Finally,  $T$  is a minimal surface of general type because it is a finite, étale quotient of the minimal surface of general type  $C_4 \times C_4$ .  $\square$

The three curves  $\Gamma_i \subset C_4 \times C_4$  are  $\xi_{xy}$ -invariant, hence their images  $\Sigma_i := t(\Gamma_i) \subset T$  are three curves isomorphic to  $C_2$  and such that  $(\Sigma_i)^2 = \frac{1}{3}(\Gamma_i)^2 = -2$ . Moreover, the curve  $\Gamma_0$  is also  $\sigma$ -invariant, whereas  $\Gamma_1$  and  $\Gamma_2$  are switched by the action of  $\sigma$ . Then  $D_S := u(\Sigma_0)$  and  $R := u(\Sigma_1) = u(\Sigma_2)$  are two disjoint curves in  $S$ , both isomorphic to  $C_2$ , such that  $(D_S)^2 = -4$  and  $R^2 = -2$ . Note that  $D_S$  is the branch locus of the double cover  $u: T \rightarrow S$ .

We can now compute the invariants of  $S$ .

**Proposition 1.5.** *The surface  $S$  is a minimal surface of general type with*

$$p_g(S) = 2, \quad q(S) = 2, \quad K_S^2 = 7.$$

*The morphism  $\beta: S \rightarrow B$  is a non-Galois triple cover, simply ramified over  $R$  and simply branched over the diagonal  $D_B \subset B$ . Finally,  $S$  contains no rational curves (in particular,  $K_S$  is ample) and contains a smooth elliptic curve, namely  $Z := \beta^*E$  (which satisfies  $Z^2 = -3$ ).*

*Proof.* We start by proving the last claim. The two smooth curves  $D_B$  and  $E$  intersect transversally at the six points corresponding to the six Weierstrass points of  $C_2$ , so  $Z := \beta^*E$  is a smooth, irreducible curve of genus 1 contained in  $S$ , such that

$$ZR = (\beta^*E).R = E.(\beta_*R) = ED_B = 6. \quad (6)$$

On the other hand,  $S$  does not contain any rational curve. Otherwise, such a curve would map onto  $E$  via  $\beta: S \rightarrow B$ , impossible because we have seen that  $\beta^*E$  is smooth of genus 1.

Since the double cover  $u: T \rightarrow S$  is branched over the curve  $D_S$ , it follows that  $D_S$  is 2-divisible in  $\text{Pic}(S)$  and moreover

$$24 = K_T^2 = 2 \left( K_S + \frac{1}{2} D_S \right)^2.$$

Using  $(D_S)^2 = -4$  and  $K_S D_S = 6$ , we find  $K_S^2 = 7$ . Since  $S$  does not contain any rational curve and  $K_S^2 > 0$ , we deduce that  $S$  is a minimal surface of general type with ample canonical class.

Now, as  $K_B = \pi^*K_A + E = E$ , the Riemann-Hurwitz formula yields

$$K_S = \beta^*K_B + R = Z + R, \quad (7)$$

and this allows us to compute  $Z^2$ . In fact, using (6) and (7), we can write

$$7 = K_S^2 = Z^2 + 2ZR + R^2 = Z^2 + 10,$$

that is  $Z^2 = -3$ .

Next, denoting by  $\chi_{\text{top}}$  the topological Euler number, we have

$$\begin{aligned} \chi_{\text{top}}(S - D_S - R) &= \frac{1}{2} \chi_{\text{top}}(T - \Sigma_0 - \Sigma_1 - \Sigma_2) \\ &= \frac{1}{2} (c_2(T) - \chi_{\text{top}}(\Sigma_0) - \chi_{\text{top}}(\Sigma_1) - \chi_{\text{top}}(\Sigma_2)) = \frac{1}{2} (12 - 3(-2)) = 9, \end{aligned}$$

so

$$c_2(S) = \chi_{\text{top}}(S) = \chi_{\text{top}}(S - D_S - R) + \chi_{\text{top}}(D_S) + \chi_{\text{top}}(R) = 9 - 2 - 2 = 5.$$

Therefore Noether's formula yields  $\chi(\mathcal{O}_S) = 1$ , that is  $p_g(S) = q(S)$ .

The existence of the surjective morphism  $\alpha: S \rightarrow A$  implies  $q \geq 2$ , and since minimal surfaces of general type with  $p_g = q \geq 3$  have either  $K^2 = 6$  or  $K^2 = 8$  (see for instance [BaCaPi06]), we deduce  $p_g(S) = q(S) = 2$ .

The morphism  $\beta: S \rightarrow B$  is a non-Galois triple cover, because  $G$  is a non-normal subgroup of index 3 in  $H$ . Since  $t: C_4 \times C_4 \rightarrow T$  is étale and  $u: T \rightarrow S$  is branched over  $D_S$ , by Lemma 1.2 it follows that  $\beta: S \rightarrow B$  is simply ramified over  $R$ , and hence simply branched over  $\beta(R) = D_B$ .  $\square$

**Remark 1.6.** The existence of surfaces  $S$  was first established in [CanFr15], using a computer-aided construction based on Magma computations. The present paper provides the first computer-free description of them. Actually,  $S$  is a *semi-isogenous mixed surface*, namely a quotient of type  $(C \times C)/G$ , where  $C$  is a smooth curve and  $G$  is a finite subgroup of  $\text{Aut}(C \times C)$ , such that the subgroup  $G^0$  of the automorphisms preserving both factors has index 2 and acts freely. In fact, with our previous notation

$$C = C_4, \quad G = \langle \xi_{xy}, \sigma \rangle, \quad G^0 = \langle \xi_{xy} \rangle.$$

The paper [CanFr15] provides a detailed study of semi-isogenous mixed surfaces, showing, among other things, that they are smooth and how to compute their invariants. For instance, [CanFr15, Proposition 2.6] allows us to prove the equality  $q(S) = 2$  without exploiting the classification of surfaces with  $p_g = q \geq 3$ .

Let us now identify the blow-up morphism  $\pi: B \rightarrow A$  with the Abel-Jacobi map

$$\text{Sym}^2(C_2) \rightarrow J(C_2).$$

If  $\Theta$  is the class of a theta divisor in  $\text{NS}(A)$ , let us define the class  $\Theta_B := \pi^*\Theta$  in  $\text{NS}(B)$ . Moreover, let us write  $x$  for the class in  $\text{NS}(B)$  given by the image of the map

$$C_2 \rightarrow \text{Sym}^2(C_2), \quad p \mapsto p_0 + p$$

where  $p_0 \in C_2$  is fixed (such a class does not depend on  $p_0$ ). Then we can prove the following

**Lemma 1.7.** *The equality  $\pi_* D_B = 4\Theta$  holds in  $\text{NS}(A)$ .*

*Proof.* This is a consequence of general results on  $g$ -fold symmetric products of curves of genus  $g$ . For instance, [Mat62, equations (1) and (5)] give in our case the relations

$$2E + D_B = 4x, \quad \Theta_B = E + x$$

in  $\text{NS}(B)$ , and these in turn imply  $D_B = 4\Theta_B - 6E$ . So the result follows by applying the push-forward map  $\pi_*: \text{NS}(B) \rightarrow \text{NS}(A)$ .  $\square$

The next step consists in describing the Albanese morphism of  $S$ .

**Proposition 1.8.** *The abelian surface  $A$  is isomorphic to  $\text{Alb}(S)$  and, up to automorphisms of  $A$ , the generically finite triple cover  $\alpha = \pi \circ \beta: S \rightarrow A$  coincides with the Albanese morphism of  $S$ . Furthermore, the only curve contracted by  $\alpha$  is  $Z$ . Finally,  $\alpha$  is branched over a divisor  $D_A$  numerically equivalent to  $4\Theta$ , having an ordinary sextuple point and no other singularities.*

*Proof.* By the universal property of the Albanese variety ([Be96, Chapter V]), the morphism  $\alpha: S \rightarrow A$  must factor through the Albanese morphism of  $S$ ; but  $\alpha$  is surjective and generically of degree 3, so it must actually coincide with the Albanese morphism of  $S$  up to automorphisms of  $A$ . Since  $\beta$  is a finite morphism,  $\alpha$  only contracts the preimage of  $E$  in  $S$ , which is  $Z$ . The branch locus  $D_A$  of  $\alpha$  is equal to the image of the diagonal  $D_B$  via  $\pi: B \rightarrow A$ ; since  $D_B$  is smooth and intersects  $E$  transversally at six points, it follows that  $D_A$  has an ordinary sextuple point and no other singularities. Finally, the fact that  $D_A$  is numerically equivalent to  $4\Theta$  follows from Lemma 1.7.  $\square$

The situation is summarized in Figure 1 below.

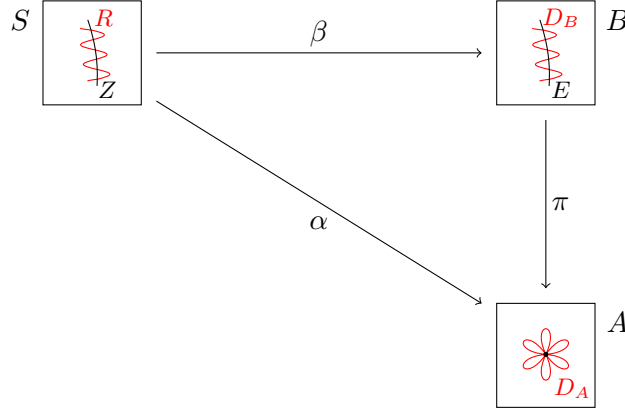


Figure 1: The triple covers  $\alpha$  and  $\beta$

Furthermore, the Stein factorization of  $\alpha: S \rightarrow A$  is described in the diagram

$$\begin{array}{ccc} S & \xrightarrow{c_Z} & \tilde{S} \\ & \searrow \alpha & \downarrow \tilde{\alpha} \\ & & A, \end{array}$$

where  $c_Z: S \rightarrow \tilde{S}$  is the birational morphism given by the contraction of the elliptic curve  $Z$ . Since  $Z^2 = -3$ , the normal surface  $\tilde{S}$  has a Gorenstein elliptic singularity of type  $\tilde{E}_6$ , see [Is14, Theorem 7.6.4].

Recall that an *irrational pencil* (or *irrational fibration*) on a smooth, projective surface is a surjective morphism with connected fibres over a curve of positive genus.

**Proposition 1.9.** *The general surface  $S$  contains no irrational pencils.*

*Proof.* Assume that  $\phi: S \rightarrow W$  is an irrational pencil on  $S$ . Since  $q(S) = 2$ , we have either  $g(W) = 1$  or  $g(W) = 2$ . On the other hand, using the embedding  $W \hookrightarrow J(W)$  and the universal property of the Albanese map, we obtain a morphism  $A \rightarrow J(W)$  whose image is isomorphic to the curve  $W$ . This rules out the case  $g(W) = 2$ , hence  $W$  is an elliptic curve and so  $A$  is a non-simple abelian surface. The proof is now complete, because  $A$  is isomorphic to the Jacobian variety  $J(C_2)$ , which is known to be simple for a general choice of  $C_2$  ([Ko76, Theorem 3.1]).  $\square$

## 2 The moduli space

A projective variety  $X$  is called of *maximal Albanese dimension* if its Albanese map  $\alpha_X: X \rightarrow \text{Alb}(X)$  is generically finite onto its image. For surfaces of general type with irregularity at least 2, this is actually a topological property, as shown by the result below.

**Proposition 2.1.** *Let  $S$  be a minimal surface of general type with  $q(S) \geq 2$ . If  $S$  is of maximal Albanese dimension, then the same holds for any surface which is homeomorphic to  $S$ . Furthermore, in the case  $q(S) = 2$  the degree of the Albanese map  $\alpha: S \rightarrow A$  is a topological invariant.*



*Proof.* This follows by the results of [Ca91], see for instance [PePol13b, Proposition 3.1].  $\square$

Proposition 2.1 allows us to study the deformations of  $S$  by relating them to those of the flat triple cover  $\beta: S \rightarrow B$ . Since the trace map provides a splitting of the short exact sequence

$$0 \rightarrow \mathcal{O}_B \rightarrow \beta_* \mathcal{O}_S \rightarrow \mathcal{E}_\beta \rightarrow 0,$$

we obtain a direct sum decomposition

$$\beta_* \mathcal{O}_S = \mathcal{O}_B \oplus \mathcal{E}_\beta, \quad (8)$$

where  $\mathcal{E}_\beta$  is a vector bundle of rank 2 on  $B$  which satisfies

$$h^0(B, \mathcal{E}_\beta) = 0, \quad h^1(B, \mathcal{E}_\beta) = 0, \quad h^2(B, \mathcal{E}_\beta) = 1 \quad (9)$$

and that, according to [Mir85], is called the *Tschirnhausen bundle* of  $\beta$ .

As in [PePol13b, Section 3] we have a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_S & \xrightarrow{d\beta} & \beta^* T_B & \longrightarrow & \mathcal{N}_\beta \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_S & \xrightarrow{d\alpha} & \alpha^* T_A & \longrightarrow & \mathcal{N}_\alpha \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{O}_Z(-Z) = \mathcal{O}_Z(-Z) & & \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \quad (10)$$

whose central column is the pullback via  $\beta: S \rightarrow B$  of a sequence

$$0 \rightarrow T_B \xrightarrow{d\pi} \pi^* T_A \rightarrow \mathcal{O}_E(-E) \rightarrow 0, \quad (11)$$

see [Sern06, p. 73]. The normal sheaf  $\mathcal{N}_\alpha$  of  $\alpha: S \rightarrow A$  is supported on the set of critical points of  $\alpha$ , namely on the reducible divisor  $R + Z$ . Analogously, the normal sheaf  $\mathcal{N}_\beta$  of  $\beta: S \rightarrow B$  is supported on the set of critical points of  $\beta$ , namely on  $R$ .

**Lemma 2.2.** *We have*

$$\mathcal{N}_\beta = (N_{R/S})^{\otimes 2} = \mathcal{O}_R(2R). \quad (12)$$

*Hence all first-order deformations of  $\beta: S \rightarrow B$  leaving  $B$  fixed are trivial.*

*Proof.* Since  $R$  is smooth, the first statement is a consequence of [Rol10, Lemma 3.2]. Furthermore, we observe that  $R^2 = -2$  implies that the line bundle  $\mathcal{N}_\beta$  has negative degree on  $R$ , hence  $H^0(R, \mathcal{N}_\beta) = 0$ . By [Sern06, Corollary 3.4.9], this shows that  $\beta: S \rightarrow B$  is rigid as a morphism with fixed target.  $\square$

Note the the last statement of Lemma 2.2 agrees with the fact that the branch locus  $D_B$  of  $\beta: S \rightarrow B$  is a rigid divisor in  $B$ .

**Lemma 2.3.** *We have*

$$h^1(S, T_S) = h^0(R + Z, \mathcal{N}_\alpha) + 1 \geq 3.$$

*Proof.* Let us write down the cohomology exact sequence associated with the short exact sequence in the central row of (10), recalling first that  $S$  is a surface of general type and therefore  $h^0(S, T_S) = 0$ :

$$0 \longrightarrow H^0(S, \alpha^*T_A) \cong \mathbb{C}^2 \longrightarrow H^0(R + Z, \mathcal{N}_\alpha) \longrightarrow H^1(S, T_S) \xrightarrow{\varepsilon} H^1(S, \alpha^*T_A)$$

Then the claim will follow if we show that  $\text{rank}(\varepsilon) = 3$ , and this can be done by using the same argument as in [PePol13b, Section 3].

More precisely, since  $T_A$  is trivial and the Albanese map induces an isomorphism  $H^1(S, \mathcal{O}_S) \cong H^1(A, \mathcal{O}_A)$ , then  $H^1(S, \alpha^*T_A) \cong H^1(A, T_A)$  and we can see  $\varepsilon$  as the map  $H^1(S, T_S) \rightarrow H^1(A, T_A)$  induced on first-order deformations by the Albanese map. By Remark 1.3 the first-order deformations of  $S$  dominate the first-order algebraic deformations of  $A$ , so  $\text{rank}(\varepsilon) \geq 3$ ; on the other hand, the Albanese variety of every deformation of  $S$  has to remain algebraic, so  $\text{rank}(\varepsilon) \leq 3$  and we are done.  $\square$

Thus, in order to understand the first-order deformations of  $S$ , we can study  $\mathcal{N}_\alpha$ .

**Lemma 2.4.** *The sheaf  $\mathcal{N}_\alpha$  is locally free of rank 1 on the reducible curve  $R + Z$ .*

*Proof.* By a standard application of Nakayama's lemma (see for instance [Kempf93, Corollary 5.3.4]), it suffices to check that the  $\mathbb{C}$ -vector space  $\mathcal{N}_{\alpha, x} / \mathfrak{m}_x \mathcal{N}_{\alpha, x}$  has dimension 1 for all  $x \in R + Z$ , where  $\mathfrak{m}_x \subset \mathcal{O}_{R+Z, x}$  is the maximal ideal. Equivalently, we will check that the vector bundle map  $d\alpha: T_S \rightarrow \alpha^*T_A$  has rank 1 at each point  $x \in R + Z$ . Let us distinguish three cases.

- If  $x \in R \setminus Z$ , then  $\alpha$  is locally of the form  $(u, v) \mapsto (u^2, v)$ , with  $x = (0, 0)$  and  $R$  given by  $u = 0$ . Then  $d\alpha$  is the linear map associated with the matrix

$$\begin{pmatrix} 2u & 0 \\ 0 & 1 \end{pmatrix},$$

which has rank 1 at the point  $x$ .

- If  $x \in Z \setminus R$ , then  $\alpha$  is locally a smooth blow-up, hence of the form  $(u, v) \mapsto (uv, v)$ , where  $x = (0, 0)$  and  $Z$  corresponds to the exceptional divisor, whose equation is  $v = 0$ . Then  $d\alpha$  is the linear map associated with the matrix

$$\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix},$$

which has rank 1 at the point  $x$ .

- Finally, if  $x \in R \cap Z$  then  $\alpha$  is locally the composition of the two maps above, so of the form  $(u, v) \mapsto (u^2v, v)$ , where  $x = (0, 0)$ , the curve  $R$  corresponds to the locus  $u = 0$  and the curve  $Z$  to the locus  $v = 0$ . Then  $d\alpha$  is the linear map associated with the matrix

$$\begin{pmatrix} 2uv & u^2 \\ 0 & 1 \end{pmatrix},$$

which has rank 1 at the point  $x$ .

This completes the proof.  $\square$

We can be more precise and compute the restrictions of  $\mathcal{N}_\alpha$  to both curves  $R$  and  $Z$ .

**Lemma 2.5.** *We have*

$$\mathcal{N}_{\alpha|Z} = \mathcal{O}_Z(-Z), \quad \mathcal{N}_{\alpha|R} = \mathcal{O}_R(2R + Z) = \mathcal{O}_R(K_R).$$

*Proof.* Let us first apply the functor  $\otimes_{\mathcal{O}_{R+Z}} \mathcal{O}_Z$  to the exact sequence forming the last column of diagram (10); using (12), we get

$$\mathcal{O}_R(2R) \otimes \mathcal{O}_Z \xrightarrow{\zeta} \mathcal{N}_{\alpha|Z} \longrightarrow \mathcal{O}_Z(-Z) \longrightarrow 0.$$

By Lemma 2.4, the sheaf  $\mathcal{N}_{\alpha|Z}$  is locally free on  $Z$ ; on the other hand,  $\mathcal{O}_R(2R) \otimes \mathcal{O}_Z$  is a torsion sheaf, hence  $\zeta$  is the zero map and so  $\mathcal{N}_{\alpha|Z} \cong \mathcal{O}_Z(-Z)$ .

Next, we apply to the same exact sequence the functor  $\otimes_{\mathcal{O}_{R+Z}} \mathcal{O}_R$ , obtaining

$$\mathcal{T} \xrightarrow{\tau} \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \quad (13)$$

Since  $\mathcal{T} := \text{Tor}_{\mathcal{O}_{R+Z}}^1(\mathcal{O}_Z(-Z), \mathcal{O}_R)$  is a torsion sheaf (supported on  $R \cap Z$ ) and  $\mathcal{O}_R(2R)$  is locally free on  $R$ , we deduce that  $\tau$  is the zero map and so (13) becomes

$$0 \longrightarrow \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \quad (14)$$

On the other hand, the curves  $R$  and  $Z$  intersect transversally at the six Weierstrass points  $p_1, \dots, p_6$  of  $R$ , so we infer

$$\mathcal{O}_Z(-Z) \otimes \mathcal{O}_R = \mathcal{O}_Z \otimes \mathcal{O}_R = \bigoplus_{i=1}^6 \mathcal{O}_{p_i}. \quad (15)$$

Hence (14) and (15) yield

$$0 \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{N}_{\alpha|R}(-2R) \longrightarrow \bigoplus_{i=1}^6 \mathcal{O}_{p_i} \longrightarrow 0,$$

that is the invertible sheaf  $\mathcal{N}_{\alpha|R}(-2R)$  has a global section whose divisor is  $\sum p_i$ . This means  $\mathcal{N}_{\alpha|R} \cong \mathcal{O}_R(2R + \sum p_i) = \mathcal{O}_R(2R + Z)$ . Finally, equation (7) shows that  $R + Z$  is a canonical divisor on  $S$ , so by using adjunction formula we obtain

$$\mathcal{O}_R(2R + Z) = \mathcal{O}_S(K_S + R) \otimes \mathcal{O}_R = \mathcal{O}_R(K_R).$$

$\square$

We can finally prove

**Proposition 2.6.** *All surfaces  $S$  constructed in Section 1 satisfy*

$$h^1(S, T_S) = 3.$$

*Proof.* By Lemma 2.3 it suffices to show the inequality  $h^0(R + Z, \mathcal{N}_\alpha) \leq 2$ . By [BHPV03, p. 62] we have a “decomposition sequence”

$$0 \longrightarrow \mathcal{O}_Z(-R) \longrightarrow \mathcal{O}_{R+Z} \longrightarrow \mathcal{O}_R \longrightarrow 0,$$

which gives, tensoring with  $\mathcal{N}_\alpha$  and using Lemma 2.5,

$$0 \longrightarrow \mathcal{O}_Z(-R - Z) \longrightarrow \mathcal{N}_\alpha \longrightarrow \mathcal{O}_R(K_R) \longrightarrow 0.$$

Since  $Z(-R - Z) = -3 < 0$ , we deduce  $H^0(Z, \mathcal{O}_Z(-R - Z)) = 0$ . So  $H^0(R + Z, \mathcal{N}_\alpha)$  injects into  $H^0(R, K_R) = \mathbb{C}^2$  and we are done.  $\square$

The moduli space of principally polarized abelian surfaces has dimension 3; moreover, the rigidity of the curve  $D_B$  in  $B$  implies that the curve  $D_A$  has only trivial deformations in  $A$ . So our surfaces  $S$  provide a 3-dimensional subset  $\mathcal{M}$  of the moduli space  $\mathcal{M}_{2,2,7}^{\text{can}}$  of (canonical models of) minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$ . Because of Proposition 2.6, the corresponding Kuranishi family is smooth; this implies that  $\mathcal{M}$  has at most quotient singularities, so it is a normal (and hence generically smooth) open subset of  $\mathcal{M}_{2,2,7}^{\text{can}}$ . In particular,  $\mathcal{M}$  provides a dense open set of a generically smooth, irreducible component of this moduli space.

Summing up, we have proven the Main Theorem stated in the introduction, namely

**Theorem 2.7.** *There exists a 3-dimensional family  $\mathcal{M}$  of minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$  such that, for all elements  $S \in \mathcal{M}$ , the canonical class  $K_S$  is ample and the Albanese map  $\alpha: S \rightarrow A$  is a generically finite triple cover of a principally polarized abelian surface  $(A, \Theta)$ , simply branched over a curve  $D_A$  numerically equivalent to  $4\Theta$  having an ordinary sextuple point and no other singularities. The family  $\mathcal{M}$  provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space  $\mathcal{M}_{2,2,7}^{\text{can}}$  of canonical models of minimal surfaces of general type with  $p_g = q = 2$  and  $K^2 = 7$ .*

**Remark 2.8.** By Proposition 2.1 the degree of the Albanese map is in our case a topological invariant, so it follows that the surfaces in  $\mathcal{M}$  lie in a different connected component of  $\mathcal{M}_{2,2,7}^{\text{can}}$  than the only other known example with the same invariants, namely the surface with  $p_g = q = 2$  and  $K^2 = 7$  constructed in [Ri15], whose Albanese map is a generically finite *double* cover of an abelian surface with polarization of type  $(1, 2)$ . Hence the family  $\mathcal{M}$  provides a substantially new piece in the fine classification of minimal surfaces of general type with  $p_g = q = 2$ .

For every surface  $S$  whose isomorphism class  $[S]$  belongs to  $\mathcal{M}$ , the normalization of the branching curve  $D_A$  of  $\alpha: S \rightarrow A$  is isomorphic to  $C_2$ , hence we obtain a morphism

$$\varsigma: \mathcal{M} \rightarrow \mathcal{M}_2, \quad \varsigma([S]) := [C_2],$$

where  $\mathcal{M}_2$  denotes as usual the coarse moduli space of curves of genus 2. Note that such a morphism is surjective by Proposition 1.1. Correspondingly, we have a morphism of deformation functors, namely

$$\delta_S: \text{Def}_S \rightarrow \text{Def}_{C_2}.$$

The next result clarifies the relation between the deformations of  $S$  and those of the curve  $C_2$ .

**Proposition 2.9.** *The following holds:*

- (1)  $\delta_S: \text{Def}_S \rightarrow \text{Def}_{C_2}$  is an isomorphism of functors.
- (2)  $\varsigma: \mathcal{M} \rightarrow \mathcal{M}_2$  is a quasi-finite morphism of degree 40.

*Proof.* (1) Since  $\dim \mathcal{M} = \dim H^1(S, T_S) = 3$ , the functor  $\text{Def}_S$  is unobstructed; moreover, the functor  $\text{Def}_{C_2}$  is clearly unobstructed, too. Proposition 1.1 implies that the first-order deformations of  $S$  dominate the first-order deformations of  $C_2$ , so the differential map

$$d\delta_S: H^1(S, T_S) \rightarrow H^1(C_2, T_{C_2}) \tag{16}$$

is surjective, and hence it is an isomorphism because  $H^1(S, T_S)$  and  $H^1(C_2, T_{C_2})$  have the same dimension. Since  $\text{Def}_S$  and  $\text{Def}_{C_2}$  are both unobstructed, this shows that  $\delta_S$  is an isomorphism of functors, see [Sern06, Corollary 2.3.7 and Remark 2.3.8].

(2) We have to show that, for each  $C_2 \in \mathcal{M}_2$ , the cardinality of  $\varsigma^{-1}(C_2)$  is at most 40 and that it is exactly 40 for a general choice of  $C_2$ .

Remark that, once  $C_2$  is fixed, the étale  $\mathbb{Z}/3\mathbb{Z}$ -cover  $c: C_4 \rightarrow C_2$  completely determines  $S$ . Conversely, we claim that, starting from  $S$ , it is possible to reconstruct the étale morphism  $c: C_4 \rightarrow C_2$  up to automorphisms of  $C_2$  and  $C_4$ . In fact, the subgroup  $\xi_{xy}$  is normal in  $H$  and the quotient  $H/\langle \xi_{xy} \rangle$  is isomorphic to  $S_3$ , hence looking at diagram (5) we see that the map

$$v \circ \gamma: T \rightarrow B = \text{Sym}^2(C_2)$$

yields the Galois closure of the triple cover  $\beta: S \rightarrow B$ . This shows that  $S$  determines the quasi-bundle  $T = (C_4 \times C_4)/\langle \xi_{xy} \rangle$ . On the other hand, since the action of  $\langle \xi_{xy} \rangle$  on  $C_4 \times C_4$  is faithful, if we know  $T$  we can reconstruct  $C_4$  and the étale  $\mathbb{Z}/3\mathbb{Z}$ -cover  $c: C_4 \rightarrow C_2$  up to automorphisms by using the rigidity result for minimal realizations of surfaces isogenous to a product proven in [Ca00, Proposition 3.13].

Summing up, the cardinality of  $\varsigma^{-1}(C_2)$  equals the number of Galois étale triple covers  $c: C_4 \rightarrow C_2$  up to equivalence. Here by “equivalence of covers” we intend commutative diagrams of the form

$$\begin{array}{ccc} C_4 & \xrightarrow{\varphi_4} & C_4 \\ c \downarrow & & \downarrow c \\ C_2 & \xrightarrow{\varphi_2} & C_2, \end{array}$$

where the horizontal arrows are automorphisms of the corresponding curves. In particular, as explained for instance in [Par91], if  $\varphi_2 = \text{id}_{C_2}$  then the number of equivalence classes of Galois triple covers  $c: C_4 \rightarrow C_2$  coincides with the number of distinct subgroups of order 3 in  $\text{Pic}^0(C_2)$ , i.e. with half the number of non-trivial 3-torsion points, that is  $(3^4 - 1)/2 = 40$ .

On the other hand, if  $C_2$  is a general curve of genus 2 its unique non-trivial automorphism is the hyperelliptic involution, which acts as the multiplication by  $-1$  on the group  $\text{Pic}^0(C_2)$  and hence trivially on the set of its 40 subgroups of order 3. Thus, for a general choice of  $C_2$ , the fibre  $\varsigma^{-1}(C_2)$  consists of exactly 40 distinct points.  $\square$

**Remark 2.10.** Let us denote by  $\mathcal{A}_2$  the coarse moduli space of principally polarized abelian surfaces. It is well-known that the Torelli map  $\tau_2: \mathcal{M}_2 \rightarrow \mathcal{A}_2$ , sending every curve to its polarized Jacobian, is an immersion, see [OS80]. Thus, composing  $\tau_2$  with  $\varsigma$ , we obtain a generically finite dominant morphism  $\tau_2 \circ \varsigma: \mathcal{M} \rightarrow \mathcal{A}_2$  of degree 40, which is the one induced by the deformations of the Albanese map  $\alpha: S \rightarrow \text{Alb}(S)$ . Observe that such a morphism is not surjective, because its image does not contain the products of elliptic curves that are not isomorphic to Jacobians.

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